PS 9

9.1

1 List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.

b) $a + b = 4$.

c) $a > b$

d) $a \mid b$

e) $\gcd(a, b) = 1$

f) $\operatorname{lcm}(a, b) = 2$

3 For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

c) $\{(2, 4), (4, 2)\}$

d) $\{(1, 2), (2, 3), (3, 4)\}$

e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

4 Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b.

b) a and b were born on the same day.

c) a has the same first name as b.

d) a and b have a common grandparent.

9.3

1 Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 1), (1, 2), (1, 3)\}$

b) $\{(1,2),(2,1),(2,2),(3,3)\}$

c) $\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

d) $\{(1,3),(3,1)\}$

2 Represent each of these relations on $\{1,2,3,4\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

b) $\{(1,1),(1,4),(2,2),(3,3),(4,1)\}$

c) $\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2)\ ,\ (3,4),(4,1),(4,2),(4,3)\}$

d) $\{(2,4),(3,1),(3,2),(3,4)\}$

5 How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?

9.5

1 Which of these relations on $\{0,1,2,3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0,0),(1,1),(2,2),(3,3)\}$

b) $\{(0,0),(0,2),(2,0),(2,2),(2,3),(3,2),(3,3)\}$

c) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$

d) $\{(0,0),(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

e) $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0)\ ,\ (2,2),(3,3)\}$

2

Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a,b) \mid a\ and\ b\ are\ the\ same\ age\}$

b) $\{(a,b) \mid a\ and\ b\ have\ the\ same\ parents\}$

c) $\{(a,b) \mid a\ and\ b\ share\ a\ common\ parent\}$

d) $\{(a,b) \mid a\ and\ b\ have\ met\}$

e) $\{(a,b) \mid a\ and\ b\ speak\ a\ common\ language\}$

3

Which of these relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1) = g(1)\}$

b) $\{(f, g) \mid f(0) = g(0)$ or $f(1) = g(1)\}$

c) $\{(f, g) \mid f(x) - g(x) = 1$ for all $x \in \mathbf{Z}\}$

d) $\{(f, g) \mid$ for some $C \in \mathbf{Z}$, for all $x \in \mathbf{Z}, f(x) - g(x) = C\}$

e) $\{(f, g) \mid f(0) = g(1)$ and $f(1) = g(0)\}$

9.6

1 Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2 Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3)$, $(3, 0), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2)$, $(1, 3), (2, 0), (2, 2), (3, 3)\}$

TEST1

1 Which ordered pairs are in the relation $\{(x, y) \mid x > y + 1\}$ on the set $\{1, 2, 3, 4\}$?

2 Consider the following relations on $\{1, 2, 3\}$.

$R_1 = \{(1, 1), (1, 3), (2, 2), (3, 1)\}$
$R_2 = \{(1, 1), (2, 2), (3, 1), (3, 3)\}$
$R_3 = \{(1, 2), (2, 1), (3, 3)\}$
$R_4 = \{(1, 3), (2, 3)\}$

(a) Which of these relations are reflexive? Justify your answers.

(b) Which of these relations are symmetric? Justify your answers.

(c) Which of these relations are antisymmetric? Justify your answers.

(d) Which of these relations are transitive? Justify your answers.

3 Find the reflexive closure and the symmetric closure of the relation $\{(1,2),(1,4),(2,3),(3,1),(4,2)\}$ on the set $\{1,2,3,4\}$

4 What is the transitive closure of the relation in problem 3?

5 (a) Show that the relation $R = \{(x,y) \mid x$ and y are bit strings containing the same number of 0 s $\}$ is an equivalence relation.

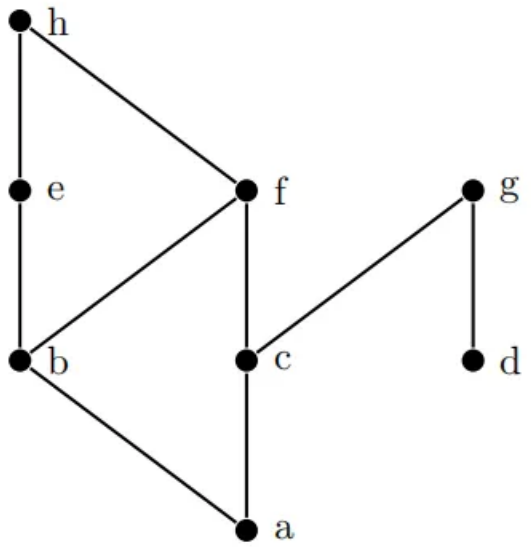
(b) What are the equivalence classes of the bit strings 1,00 , and 101 under the relation R ?

6 (a) Are the sets $\{1,3,6\},\{2,4,7\},and\ \{5\}\ a\ partition\ of \{1,2,3,4,5,6,7\}$

(b) Are the sets $\{1,2,4,5\},\{3,6,7\},and\ \{2,3\}\ a\ partition\ of \{1,2,3,4,5,6,7\}$?

7 Show that the inclusion relation,
$\{(A,B) \mid A \subseteq B\}, is\ a\ partial\ ordering\ on\ the\ set\ of\ all\ subsets\ of\ \mathbf{Z}.$

8 What are the minimal and maximal elements in the poset with the following Hasse diagram? Are there least and greatest elements?



TEST

1

- Consider the following relations on the set of positive integers.

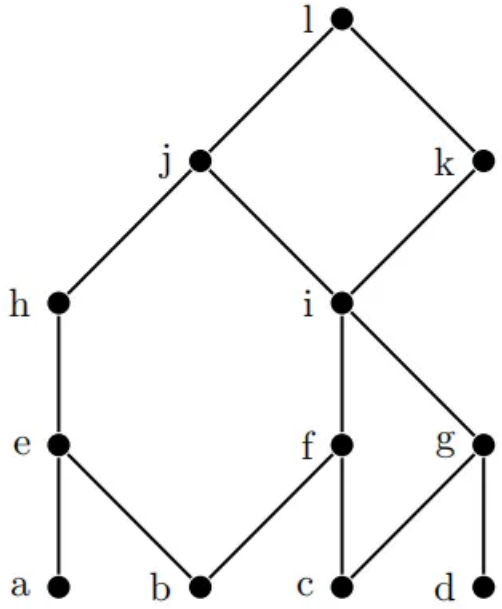
$$R_1 = \{(x, y) \mid x + y > 10\}$$
$$R_2 = \{(x, y) \mid y \text{ divides } x\}$$
$$R_3 = \{(x, y) \mid \gcd(x, y) = 1\}$$
$$R_4 = \{(x, y) \mid x \text{ and } y \text{ have the same prime divisors }\}$$

- (a) Which of these relations are reflexive? Justify your answers.
  (b) Which of these relations are symmetric? Justify your answers.
  (c) Which of these relations are antisymmetric? Justify your answers.
  (d) Which of these relations are transitive? Justify your answers

2 Suppose that R_1 and R_2 are symmetric relations on a set A. Prove or disprove that $R_1 - R_2$ is also symmetric.

4 Show that the relation $R = \{(x, y) \mid x - y \; is \; an \; integer\}$ is an equivalence relation on the set of rational numbers. What are the equivalence classes of 0 and $\frac{1}{2}$?

5 Consider the poset with the following Hasse diagram.



(a) Find all maximal elements of the poset.

(b) Find all minimal elements of the poset.

(c) Find the least element of the poset if it exists, or show that it does not exist.

(d) Find the greatest element of the poset if it exists, or show that it does not exist.

(e) What is the greatest lower bound of the set $\{a, b, c\}$?

(f) What is the least upper bound of the set $\{a, b, c\}$?

6 Use a topological sort to order the elements of the poset with the Hasse diagram given in problem 5.